LESSON 13

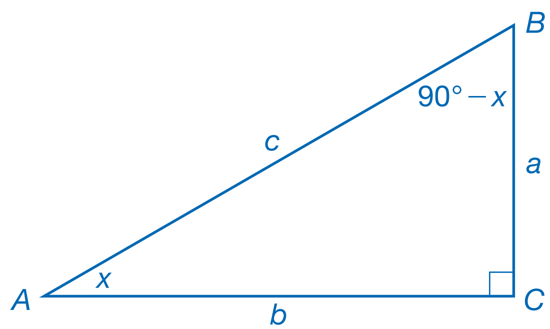
Relationships between Trigonometric Functions

UNDERSTAND Sine, cosine, and tangent are trigonometric **functions**. The **input** of each function is an angle measure. For each trigonometric function, every acute angle measure produces a different **output**, or value of the function. The values change in a predictable way over the domain $0° < x < 90°$.

Function	Relationship between Inputs and Outputs
$f(x) = \sin x$	As x increases, the value of the sine, $f(x)$, increases.
$f(x) = \cos x$	As x increases, the value of the cosine, $f(x)$, decreases.
$f(x) = \tan x$	As x increases, the value of the tangent, $f(x)$, increases.

UNDERSTAND The sum of the measures of the interior angles of a triangle is 180°. Every right triangle has one right angle, so the sum of the measures of the two acute angles in any right triangle must be equal to $(180 - 90)°$, or 90°. Angles that add up to 90° are **complementary angles**.

In $\triangle ABC$ below, the degree measure of $\angle A$ is x. Since $\angle B$ is the complement of $\angle A$, the degree measure of $\angle B$ is $90° - x$. Compare the trigonometric ratios for the two angles.



$\sin x = \frac{a}{c}$ $\quad$ $\cos x = \frac{b}{c}$ $\quad$ $\tan x = \frac{a}{b}$

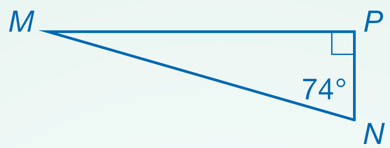
$\sin (90° - x) = \frac{b}{c}$ $\quad$ $\cos (90° - x) = \frac{a}{c}$ $\quad$ $\tan (90° - x) = \frac{b}{a}$

In a right triangle, the opposite leg for one acute angle is the adjacent leg for the other acute angle. So, the sine of one acute angle is equal to the cosine of its complement, and vice versa. The tangent of an acute angle is the reciprocal of the tangent of its complement. These relationships are summarized below.

$$\sin x = \cos (90° - x) \qquad \cos x = \sin (90° - x) \qquad \tan x = \frac{1}{\tan (90° - x)}$$

Connect

In $\triangle MNP$, $\angle N$ measures 74° and sine of $\angle N$ is approximately 0.96. What is the measure of $\angle M$ and the cosine of $\angle M$?



1 Find m$\angle M$.

The acute angles of a right triangle are complementary.

$m\angle M + m\angle N = 90°$

$m\angle M = 90° - m\angle N$

$m\angle M = 90° - 74°$

▶ $m\angle M = 16°$

2 Find cos M.

The cosine of an angle is equal to the sine of its complement.

$\cos M = \sin N$

▶ $\cos M \approx 0.96$

TRY

The tangent of $\angle N$ is approximately 3.5. What is the approximate value of the tangent of $\angle M$?

EXAMPLE A Make a chart showing the sine, cosine, and tangent values for angle measures in the domain {20, 40, 60, 80}. Analyze the values and describe how the outputs change as the inputs change.

1 Make a chart and use a calculator to approximate trigonometric ratios.

To estimate the sine of 20° by using your calculator, press SIN, enter 20, and press ENTER.

Use the COS key to estimate cosine and the TAN to estimate tangent.

Repeat this process for 40°, 60°, and 80°.

Measure of Angle	Sine of Angle	Cosine of Angle	Tangent of Angle
20°	0.34	0.94	0.36
40°	0.64	0.77	0.84
60°	0.87	0.5	1.73
80°	0.98	0.17	5.67

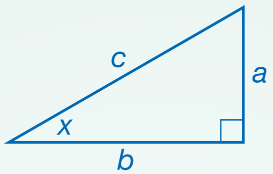
2 Analyze the chart.

As the angle measures increase, the sine values and tangent values also increase. However, as the angle measures increase, the cosine values decrease.

TRY

Use a calculator to determine the values of the sine, cosine, and tangent for the inputs 0° and 90°.

EXAMPLE B Use the triangle below to prove that $\tan x = \frac{\sin x}{\cos x}$.



1 Write trigonometric ratios for x.

The side opposite the angle has length a. The side adjacent to the angle has length b.

The hypotenuse has length c.

$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos x = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan x = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$

2 Divide the ratios for sine and cosine.

Remember that dividing by a fraction is the same as multiplying by its reciprocal.

$$\frac{\sin x}{\cos x} = \frac{\frac{a}{c}}{\frac{b}{c}}$$

$$\frac{\sin x}{\cos x} = \frac{a}{c} \cdot \frac{c}{b}$$

$$\frac{\sin x}{\cos x} = \frac{a}{b}$$

This is identical to the ratio for the tangent.

$$\frac{\sin x}{\cos x} = \tan x$$

DISCUSS

If $\sin y = \frac{4}{5}$ and $\cos y = \frac{3}{5}$, what is the value of $\tan y$? Is there only one way to determine this answer?

Practice

Let x be the degree measure of an acute angle in a right triangle. Fill in each blank with an equivalent expression containing a trigonometric function with x as the input.

1. $\cos(90° - x) =$ ____________

2. $\sin(90° - x) =$ ____________

3. $\tan(90° - x) =$ ____________

4. $\dfrac{\cos(90° - x)}{\sin(90° - x)} =$ ____________

REMEMBER If x is the measure of one acute angle in a right triangle, the other angle measures $90° - x$.

Choose the best answer.

5. Which equation is true?

A. $\cos 48° = \sin 42°$

B. $\cos 48° = \sin 48°$

C. $\cos 48° = \cos 42°$

D. $\tan 48° = \tan 42°$

6. Which equation is true?

A. $\tan 67° = \tan 23°$

B. $\tan 67° = \dfrac{1}{\tan 23°}$

C. $\tan 67° = \dfrac{\cos 67°}{\sin 67°}$

D. $\tan 67° = \dfrac{\sin 23°}{\cos 23°}$

7. Which two angles are complementary angles?

A. 95° and 85°

B. 12° and 78°

C. 36° and 36°

D. 90° and 45°

8. If $\sin 14.25° \approx \frac{16}{65}$ and $\cos 14.25° \approx \frac{63}{65}$, what is the approximate value of $\tan 14.25°$?

A. 65

B. $\frac{63}{16}$

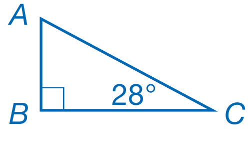
C. $\frac{16}{63}$

D. $\frac{16}{65}$

HINT Division is equivalent to multiplication by the reciprocal.

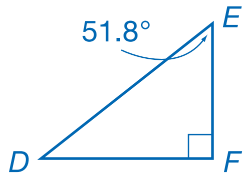
Use the given information to find the indicated measures.

9. $\sin 28° \approx 0.47$



$m\angle A =$ __________ $\cos A =$ __________

10. $\cos 51.8° \approx 0.62$ $\sin 51.8° \approx 0.79$



$m\angle D =$ __________ $\cos D =$ __________

Complete the chart and use it to answer questions 12 and 13.

11. Use a calculator to complete the chart below. Round to the nearest hundredth.

Measure of Angle	Sine of Angle	Cosine of Angle	Tangent of Angle
15°			
35°	0.57		
55°			1.43
75°		0.26	

12. Would you expect tan 70° to be less than or greater than 1.43? Explain.

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13. If $\cos x \approx 0.89$, the value of x must lie between what two angle measures?

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Solve without using a calculator.

14. EXPLAIN Given that $\sin 26.45° \approx 0.45$ and $\sin 63.55° \approx 0.90$, find the approximate value of tan 26.45°. Explain how you found your answer.

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15. PROVE Follow the given steps and use the side lengths of the triangle below to prove the Pythagorean Identity: $\sin^2 x + \cos^2 x = 1$.

Use side lengths a, b, and c to write trigonometric ratios.	$\sin x = \frac{a}{c}$ $\cos x = \frac{\;}{\;}$
Write the formula for the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Divide both sides of that equation by c^2.	$\frac{a^2}{c^2} + \frac{\;}{\;} = \frac{c^2}{c^2}$
Rewrite the expressions on the left side as ratios squared. Simplify the right side.	$\left(\frac{a}{c}\right)^2 + \left(\frac{\;}{\;}\right)^2 =$ ____________
Substitute sin x and cos x for the ratios in the equation to show the Pythagorean Identity.	____________________